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The ADER path to constructing very high-order schemes for approximating hyperbolic equations

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Sixty years ago, the Russian mathematician Sergei Godunov introduced his method to solve the Euler equations of gas dynamics, thus creating the Godunov school of thought for the numerical approximation of hyperbolic conservation laws [1]. The building block of the original first-order Godunov upwind method is the conventional piece-wise constant data Riemann problem.

ADER is a fully discrete approach to construct high-order extensions of the Godunov first-order method, in which the conventional Riemann problem is replaced by the generalized Riemann problem (GRPk), a piece-wise smooth data Cauchy problem in which the equations may include stiff source terms [2], [3],[4]. The ADER methodology operates in both the finite volume and the discontinuous Galerkin finite element frameworks [5]. In the finite volume formulation, the initial condition for each local GRPk consists of non-linear reconstruction polynomials of arbitrary degree k. The schemes extend to systems with stiff or non-stiff source terms, in multiple space dimensions, on structured or unstructured meshes. The resulting schemes are of arbitrary $k+1$ order accuracy in both space and time; there is no theoretical accuracy barrier. There are by now several methods for solving the generalized Riemann problem, thus giving rise to several classes of ADER schemes. See for example [6], [7] and [8]. In numerous applications performed over the years it has been established that these schemes are orders-of-magnitude cheaper than low-order methods for attaining a prescribed, small error and are therefore mandatory for ambitious scientific and technological applications.

Here I review some key aspects of the ADER methodology and discuss its strengths, shortcomings and issues of current research interest. Sample applications will be shown.

References

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